Journal of Statistical Physics, Vol. 67, Nos. 5/6, 1992

The Number and Size of Branched Polymers in High Dimensions

Takashi Hara[1] **and Gordon Slade**[2]

Received November 26, 1991

We consider two models of branched polymers (lattice trees) on the d-dimensional hypercubic lattice: (i) the nearest-neighbor model in sufficiently high dimensions, and (ii) a "spread-out" or long-range model for $d > 8$, in which trees are constructed from bonds of length less than or equal to a large parameter L. We prove that for either model the critical exponent θ for the number of branched polymers exists and equals 5/2, and that the critical exponent ν for the radius of gyration exists and equals 1/4. This improves our earlier results for the corresponding generating functions. The proof uses the lace expansion, together with an analysis involving fractional derivatives which has been applied previously to the self-avoiding walk in a similar context.

KEY WORDS: Branched polymers; lattice trees; lattice animals; lace expansion; critical exponents.

1. THE MODELS AND RESULTS

Recently[1] we used the lace expansion to prove that the lattice tree and lattice animal critical exponents for the susceptibility and correlation length of order two exist and take their mean-field values $\gamma = 1/2$ and $\nu = 1/4$ in two situations: (i) for the usual nearest-neighbor bond models on the hypercubic lattice $\mathbf{Z}^d$ if the dimension d is sufficiently high, and (ii) for "spread-out" or long-range models (defined below) if $d > 8$. We now consider the more detailed question of the existence of the critical exponents for the number of n-bond trees and the radius of gyration. Mathematically this question asks for the large-n asymptotics of a

[1] Department of Applied Physics, Tokyo Institute of Technology, Oh-Okayama, Meguro-ku, Tokyo 152, Japan.

[2] Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1.

0022-4715/92/0600-1009$06.50/0

sequence, given the behavior of its generating function near the closest singularity to the origin.

To simplify the discussion, we restrict attention to lattice trees, although we expect that a similar analysis can be carried out for lattice animals in the above two situations. Our results are that for the above two models the exponent θ for the number of n-bond trees exists and equals 5/2, and the exponent ν for the radius of gyration exists and equals 1/4. The method of proof uses the lace expansion of refs. 1 and 2, together with the "fractional derivative" analysis used in ref. 3 to study the analogous problem for the self-avoiding walk. However, a new technical difficulty arises in the application of these ideas to branched polymers, due to a delicate cancellation in our expression for the amplitude in the asymptotic formula for the number of n-bond trees.

The basic definitions are as follows. A nearest-neighbor bond in $\mathbf{Z}^d$ is an unordered pair of sites in $\mathbf{Z}^d$ separated by unit Euclidean distance. A nearest-neighbor tree is a connected bond cluster with no closed loops, constructed from nearest-neighbor bonds. Spread-out trees are connected bond clusters with no closed loops, constructed from bonds given by pairs of distinct sites $x, y \in \mathbf{Z}^d$ with $\|x-y\|_\infty \leqslant L$, where L is a parameter which will be taken to be large depending on the dimension. Although trees are defined to be sets of bonds, we will say that a tree T contains a site x if x is an endpoint of a bond in T. For $x \in \mathbf{Z}^d$, we denote by $t_n(x)$ the number of trees consisting of n bonds [and hence $(n+1)$ sites] which contain both x and the origin. For $n=0$ we set $t_0(x)=\delta_{0,x}$. Let $t_n = \sum_x t_n(x) = (n+1) \cdot \#\{n\text{-bond trees containing the origin}\}$, and let $a_n = t_n/(n+1)^2$ be the number of n-bond trees modulo translation. It is known[4] from a subadditivity argument that the limit

$$\lambda = \lim_{n \to \infty} a_n^{1/n} = \sup_{n \geqslant 1} a_n^{1/n} \tag{1.1}$$

exists and is nonzero and finite, and it is widely believed[5] that in all dimensions there are constants A, θ such that

$$a_n \sim A \lambda^n n^{-\theta} \tag{1.2}$$

(with a logarithmic correction for $d=8$). Here the symbol $\sim$ means that the ratio of the left and right sides approaches unity as n goes to infinity. The best general rigorous bounds are $\lambda^n n^{-O(\log n)} \leqslant a_n \leqslant \lambda^n$; the upper bound is given by (1.1) and the lower bound is obtained in [ref. 6]. The critical exponent θ is believed to depend only on the dimension, and in particular to be the same for both the nearest-neighbor and spread-out models, for any $L \geqslant 1$. The mean radius of gyration R_n is given by

$$R_n^2 = \frac{1}{2t_n} \sum_x |x|^2 \, t_n(x) = \frac{1}{t_n} \sum_{T \ni 0:\, |T| = n} \; \sum_{x \in T} |x - \bar{x}_T|^2 \tag{1.3}$$

where $|x|$ denotes the Euclidean length of x, $\bar{x}_T = (n+1)^{-1} \sum_{x \in T} x$ is the center of mass of T, and $|T|$ denotes the number of bonds in T. The radius of gyration is believed to satisfy

$$R_n \sim D n^{\nu} \tag{1.4}$$

for some D and ν, with ν depending only on the dimension (again with a logarithmic correction for $d = 8$). See [ref. 7] for a recent Monte Carlo analysis of ν, including dimensions $d = 8, 9$.

The main result of this paper is the following theorem.

Theorem 1.1. For the nearest-neighbor model in sufficiently high dimensions, or for the spread-out model with $d > 8$ and $L \geqslant L_0$ for some sufficiently large $L_0 = L_0(d)$, there are positive A, D such that:

(a) $a_n = A\lambda^n n^{-5/2}[1 + O(n^{-\varepsilon})]$ for any $\varepsilon < \min\{1/2, (d-8)/4\}$.

(b) $R_n = D n^{1/4}[1 + O(n^{-\varepsilon})]$ for any $\varepsilon < \min\{1/2, (d-8)/4\}$.

The point of departure of the proof is the results obtained in ref. 1 for the corresponding generating functions. These generating functions are defined in terms of the *two-point function*, which is given by

$$G_z(x) = \sum_{n=0}^{\infty} t_n(x) \, z^n \tag{1.5}$$

The *susceptibility* is then defined by

$$\chi(z) = \sum_{x \in \mathbf{Z}^d} G_z(x) = \sum_{n=0}^{\infty} t_n z^n \tag{1.6}$$

and the *correlation length of order two* is defined by

$$\xi_2(z) = \left[\frac{1}{\chi(z)} \sum_x |x|^2 \, G_z(x) \right]^{1/2} = \left[2 \, \frac{\sum_n \sum_{T \ni 0, |T| = n} (n+1) \, R_n^2 z^n}{\sum_n \sum_{T \ni 0, |T| = n} (n+1) \, z^n} \right]^{1/2} \tag{1.7}$$

Let $z_c = \lambda^{-1}$. It was proved in ref. 1 that for the nearest-neighbor model above some dimension d_0, and for the spread-out model with $d > 8$ and L sufficiently large,

$$\chi(z) \approx (z_c - z)^{-1/2} \tag{1.8}$$

and

$$\xi_2(z) \approx (z_c - z)^{-1/4} \tag{1.9}$$

where $f(z) \approx g(z)$ means that there are positive constants c_1, c_2 such that $c_1 g(z) \leqslant f(z) \leqslant c_2 g(z)$ uniformly in positive $z < z_c$. [In this paper (1.8) and (1.9) are improved to asymptotic formulas, with bounds on the errors.]

To prove Theorem 1.1(a), we begin by writing

$$\chi(z) = \frac{B}{(z_c - z)^{1/2}} + \mathscr{E}(z) \tag{1.10}$$

The lace expansion was used in ref. 1 to obtain a formula for $\chi(z)$, in terms of which B and $\mathscr{E}(z)$ can be identified. We do this in the next section, using the notation and ideas of ref. 1. We then apply Lemma 3.3 of ref. 3, which states in particular the following.

Lemma 1.2. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ have radius of convergence greater than or equal to $R > 0$. If for some $b \geqslant 1$ the bound $|f'(z)| \leqslant \text{const} \cdot |R - z|^{-b}$ holds uniformly in $|z| < R$, then $|c_n| \leqslant O(R^{-n} n^{-\alpha})$ for any $\alpha < 2 - b$.

The coefficient of z^n in the series expansion of $(z_c - z)^{-1/2}$ is asymptotic to a multiple of $\lambda^n n^{-1/2}$, with a correction of order $\lambda^n n^{-3/2}$. If it could be shown that for $\varepsilon < \min\{1/2, (d-8)/4\}$

$$|\mathscr{E}'(z)| \leqslant \text{const} \cdot |z_c - z|^{\varepsilon - 3/2} \tag{1.11}$$

uniformly in $|z| < z_c$, then by Lemma 1.2 it would follow that the coefficient of z^n in $\mathscr{E}(z)$ is $O(z_c^{-n} n^{-\alpha})$ for every $\alpha < \varepsilon + 1/2$, and hence by (1.10) that $t_n = A\lambda^n[n^{-1/2} + O(n^{-\alpha})]$, which gives Theorem 1.1(a). Thus, Theorem 1.1(a) follows from (1.11). In the next section we reduce the proof of (1.11) to some basic estimates.

The amplitude A can be calculated easily in terms of B. Also, integration of (1.11) gives the bound

$$|\mathscr{E}(z)| \leqslant \text{const} \cdot |z_c - z|^{\varepsilon - 1/2} \tag{1.12}$$

for the error term in (1.10).

2. REDUCTION OF THE PROOF TO SOME BASIC ESTIMATES

In this section we prove Theorem 1.1, given the bounds of Lemma 2.1 below. To state the lemma, we first need to recall some definitions and notation from refs. 1 and 3.

Our starting point is the following formula for the susceptibility, which is given in (3.8) of ref. 1:

$$\chi(z) = \frac{\hat{h}_z(0)}{\hat{F}_z(0)}, \qquad \hat{F}_z(0) \equiv 1 - \Omega z \hat{h}_z(0) \tag{2.1}$$

Here Ω is the coordination number, given by $\Omega = 2d$ for the nearest-neighbor model and $\Omega = (2L+1)^d - 1$ for the spread-out model. The function $\hat{h}_z(k)$ is given by

$$\hat{h}_z(k) = g_z + \hat{\Pi}_z(k) \tag{2.2}$$

where

$$g_z = \sum_{T \ni 0} z^{|T|} \tag{2.3}$$

and $\hat{\Pi}_z(k)$ is defined in Section 2.1.1 of ref. 1. It is shown in ref. 1 that for the spread-out model with $d > 8$ and L sufficiently large,

$$g_z \leqslant 4 \qquad \text{and} \qquad \hat{\Pi}_z(k) \leqslant O(L^{1-d}) \tag{2.4}$$

uniformly in $|z| \leqslant z_c$. It is also shown that $z_c \leqslant O(L^{-d})$. Similar bounds can be obtained for the nearest-neighbor model in sufficiently high dimensions. However, to simplify the exposition, we restrict attention in the remainder of this paper to the spread-out model; analogous considerations hold for the nearest-neighbor model.

We also use the following definition of "fractional derivatives." Given a power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ and $\varepsilon > 0$, we define

$$\delta_z^{\varepsilon} f(z) = \sum_{n=1}^{\infty} n^{\varepsilon} c_n z^n \tag{2.5}$$

We define a norm on power series by setting

$$\|f(z)\| = \sum_{n=0}^{\infty} |c_n| \, |z^n|$$

This norm satisfies $\|f(z)\, g(z)\| \leqslant \|f(z)\| \, \|g(z)\|$. Suppose that the radius of convergence R of $f(z)$ is finite and nonzero. For $\varepsilon \in (0, 1)$ it is proved in Lemma 3.2 of ref. 3 that for $|z| \leqslant R$

$$|f(z) - f(R)| \leqslant 2^{1-\varepsilon} \|\delta_z^{\varepsilon} f(R)\| \, |1 - z/R|^{\varepsilon} \tag{2.6}$$

The following lemma will be proved in Section 3.

Lemma 2.1. For any $d>8$ there is an L_0 such that for $L \geqslant L_0$ the following hold.

(i) $|\hat{F}_z(0)| \leqslant K\,|1-z/z_c|^{1/2}$ for all $|z| \leqslant z_c$, with K independent of L.

(ii) $\|\delta_z^\gamma[z\hat{h}_z(0)]\,|_{z_c}\| \leqslant K\Omega^{-1}$, for any $\gamma < 1/2$, with K independent of L (but dependent on γ).

(iii) For any z with $|z| \leqslant z_c$ and for any $\varepsilon < \min\{1/2, (d-8)/4\}$,

$$\frac{d}{dz}(z\hat{\Pi}_z(k)) = [z\Omega\chi(z)+1]\;\hat{\Psi}_z(k) \tag{2.7}$$

where

$$z\Omega\hat{\Psi}_z(0) = b + \beta(z)$$

with $\lim_{L\to\infty} b = 0$, $|\beta(z)| \leqslant K_1\,|1-z/z_c|^\varepsilon$ with K_1 independent of L (but dependent on ε), and $\lim_{L\to\infty} \hat{\Psi}_z(0) = 0$.

(iv) For any $\varepsilon < \min\{1/2, (d-8)/4\}$, $\|\delta_z^\varepsilon \nabla_k^2 \hat{\Pi}_{z_c}(0)\| < KL^2$, with K independent of L (but dependent on ε).

(v) For any z with $|z| \leqslant z_c$ and for any $\varepsilon < \min\{1/2, (d-8)/4\}$,

$$\left\| \frac{d}{dz} \nabla_k^2 \hat{\Pi}_z(0) \right\| \leqslant KL^{2+d}\,|1-z/z_c|^{\varepsilon-1}$$

with K independent of L (but dependent on ε).

The following bounds are immediate consequences of (2.6) and Lemma 2.1(ii), (iv). For any $|z| \leqslant z_c$,

$$|z_c\hat{h}_{z_c}(0) - z\hat{h}_z(0)| \leqslant C_1\Omega^{-1}\,|1-z/z_c|^\varepsilon \tag{2.8}$$

and

$$|\nabla_k^2\hat{\Pi}_{z_c}(0) - \nabla_k^2\hat{\Pi}_z(0)| \leqslant C_2 L^2\,|1-z/z_c|^\varepsilon \tag{2.9}$$

where C_1 and C_2 are independent of L.

2.1. Identification of B and $\mathscr{E}$

In this section we identify the constant B and function $\mathscr{E}$ of (1.10). We will use the fact that

$$\frac{d}{dz}(zg_z) = \chi(z) \tag{2.10}$$

which can be seen from (2.3) and (1.6).

It follows from results of ref. 1 that $\hat{F}_z(0)$ is analytic in $|z|<z_c$ and continuous on $|z|\leqslant z_c$. Because $\chi(z)$ diverges at z_c, it follows that $\hat{F}_{z_c}(0)=0$. By (2.1), (2.2), (2.10), and (2.7),

$$\frac{d\hat{F}_z(0)}{dz} = -\Omega\left\{\chi(z)+\frac{d}{dz}\left[z\hat{\Pi}_z(0)\right]\right\} = -\Omega\chi(z)[1+z\Omega\hat{\Psi}_z(0)]-\Omega\hat{\Psi}_z(0) \tag{2.11}$$

Therefore

$$-2\hat{F}_z(0)\frac{d\hat{F}_z(0)}{dz}=2\Omega\hat{h}_z(0)[1+z\Omega\hat{\Psi}_z(0)]+2\Omega\hat{F}_z(0)\,\hat{\Psi}_z(0) \tag{2.12}$$

Integrating (2.12) along a contour joining z to z_c and using the fact that $\hat{F}_{z_c}(0)=0$ gives

$$\hat{F}_z(0)^2=2\Omega\int_z^{z_c}\{\hat{h}_z(0)[1+z\Omega\hat{\Psi}_z(0)]+\hat{F}_z(0)\,\hat{\Psi}_z(0)\}\,dz \tag{2.13}$$

By (2.13),

$$\hat{F}_z(0)^2=B_1^2(z_c-z)+E(z) \tag{2.14}$$

where $B_1^2=2\Omega(1+b)\,\hat{h}_{z_c}(0)$ (b is the constant of Lemma 2.1(iii)) and $E(z)=E_1(z)+E_2(z)+E_3(z)$, with

$$E_1(z)=2\Omega\int_z^{z_c}[\hat{h}_z(0)-\hat{h}_{z_c}(0)][1+z\Omega\hat{\Psi}_z(0)]\,dz \tag{2.15}$$

$$E_2(z)=2\Omega\hat{h}_{z_c}(0)\int_z^{z_c}[z\Omega\hat{\Psi}_z(0)-b]\,dz \tag{2.16}$$

and

$$E_3(z)=2\Omega\int_z^{z_c}\hat{F}_z(0)\,\hat{\Psi}_z(0)\,dz \tag{2.17}$$

This gives

$$\frac{1}{\hat{F}_z(0)}=\frac{1}{B_1(z_c-z)^{1/2}}+\mathscr{E}_1(z) \tag{2.18}$$

with

$$\mathscr{E}_1(z)=\frac{1-[1+B_1^{-2}(z_c-z)^{-1}\,E]^{1/2}}{[B_1^2(z_c-z)+E]^{1/2}} \tag{2.19}$$

Therefore

$$\chi(z) = \frac{\hat{h}_z(0)}{\hat{F}_z(0)} = \frac{B}{(z_c - z)^{1/2}} + \mathscr{E}(z) \tag{2.20}$$

with

$$B = \frac{\hat{h}_{z_c}(0)}{B_1} = \left[\frac{\hat{h}_{z_c}(0)}{2\Omega(1+b)} \right]^{1/2} \tag{2.21}$$

and

$$\mathscr{E}(z) = \frac{\hat{h}_z(0) - \hat{h}_{z_c}(0)}{B_1(z_c - z)^{1/2}} + \hat{h}_z(0)\, \mathscr{E}_1(z) \tag{2.22}$$

2.2. Proof of Theorem 1.1(a) Given Lemma 2.1

As observed at the end of Section 1, to prove Theorem 1.1(a), it suffices to show that $|\mathscr{E}'(z)| \leqslant \text{const} \cdot |z_c - z|^{\varepsilon - 3/2}$ for all $|z| \leqslant z_c$ (the constant can depend on L). By (2.22), we have

$$\mathscr{E}'(z) = \frac{\hat{h}'_z}{B_1(z_c - z)^{1/2}} + \frac{\hat{h}_z - \hat{h}_{z_c}}{2B_1(z_c - z)^{3/2}} + \hat{h}'_z \mathscr{E}_1(z) + \hat{h}_z \mathscr{E}'_1(z) \tag{2.23}$$

It is routine to bound each term on the right side, given the following lemma.

Lemma 2.2. For any $d > 8$ there is a $c > 0$ (independent of L) and an $L_0 \geqslant 0$ such that for $L \geqslant L_0$

$$|\hat{F}_z(0)| \geqslant c\, |1 - z/z_c|^{1/2} \tag{2.24}$$

for any z with $|z| \leqslant z_c$.

Proof. Let $w = z/z_c$; we are interested in $|w| \leqslant 1$. We prove (2.24) separately for w in a neighborhood of 1 and outside of this neighborhood.

Beginning with w near 1, we first observe that since $\lim_{L \to \infty} b = 0$, $\hat{\Pi}_z(0) \leqslant O(L^{1-d})$ and $g_{z_c} \geqslant 1$, we have $B_1^2 \geqslant \Omega$ for L sufficiently large. Hence by (2.14)

$$|\hat{F}_z(0)^2| \geqslant \Omega\, |z_c - z| - |E(z)| \tag{2.25}$$

Lemma 2.1 can be used in conjunction with (2.6) to show that for $\varepsilon < \min\{1/2, (d-8)/4\}$ there is a constant K, independent of L, such that

$$|E(z)| \leqslant \Omega z_c K\, |1 - z/z_c|^{1+\varepsilon} \tag{2.26}$$

Since $\hat{F}_{z_c}(0)=0$ and $1/2 \leqslant \hat{h}_{z_c}(0) \leqslant 9/2$, we have

$$\frac{1}{5z_c} \leqslant \frac{2}{9z_c} \leqslant \Omega = \frac{1}{z_c \hat{h}_{z_c}(0)} \leqslant \frac{2}{z_c} \tag{2.27}$$

Using (2.26) and (2.27) in (2.25) gives

$$|\hat{F}_z(0)^2| \geqslant \frac{1}{5} |1-w| (1 - 10K |1-w|^{\varepsilon})$$

This gives

$$|\hat{F}_z(0)| \geqslant \frac{1}{\sqrt{10}} |1-w|^{1/2} \tag{2.28}$$

for z such that $|w| \leqslant 1$ and $|1-w| \leqslant (20K)^{-1/\varepsilon}$.

Let U denote the closed unit disk in the w-plane, and $V = \{w \in U: |1-w| \leqslant (20K)^{-1/\varepsilon}\}$. It remains to obtain the bound (2.24) on the set $U \backslash V$. For this it suffices to show that on the boundary of $U \backslash V$, $|\hat{F}_z(0)| \geqslant c'$ for some positive constant c' which does not depend on L. In fact, given such a bound it would then follow from the maximum modulus principle applied to $1/\hat{F}_z(0)$ (which would be analytic in w in the interior of $U \backslash V$ and continuous on its boundary) that for $w \in U \backslash V$,

$$|\hat{F}_z(0)| \geqslant c' = \frac{c'}{|1-w|^{1/2}} |1-w|^{1/2} \geqslant \frac{c'}{\sqrt{2}} |1-w|^{1/2} \tag{2.29}$$

A positive L-independent lower bound on $\hat{F}_z(0)$, for w in the intersection of the boundaries of $U \backslash V$ and V, follows immediately from (2.28). The remainder of the proof gives such a bound for the remainder of the boundary of $U \backslash V$, i.e., for $w = e^{i\theta}$ with $\theta \in [\theta_0, 2\pi - \theta_0]$, for some positive θ_0 which is independent of L.

By (2.1) with $z = z_c e^{i\theta}$,

$$|\hat{F}_z(0)| \geqslant 1 - z_c \Omega \, |g_z| - z_c \Omega \, |\hat{\Pi}_z(0)| \tag{2.30}$$

We define $\tilde{g}_z$ by subtracting the two lowest-order terms in g_z:

$$\tilde{g}_z = g_z - 1 - \Omega z \tag{2.31}$$

Then $\tilde{g}_z$ is a power series in z with positive coefficients, and hence

$$|\tilde{g}_z| \leqslant \tilde{g}_{z_c} \tag{2.32}$$

for $|z| \leqslant z_c$. Direct calculation, for $z = z_c e^{i\theta}$, gives

$$|g_{z_c}|^2 - |g_z|^2 = |\tilde{g}_{z_c}|^2 - |\tilde{g}_z|^2 + 2(\tilde{g}_{z_c} - \mathrm{Re}\,\tilde{g}_z) + 2\Omega z_c(1-\cos\theta) + 2\Omega z_c(\tilde{g}_{z_c} - \cos\theta\,\mathrm{Re}\,\tilde{g}_z - \sin\theta\,\mathrm{Im}\,\tilde{g}_z) \tag{2.33}$$

Using (2.32), we conclude that

$$|g_{z_c}|^2 - |g_z|^2 \geqslant 2\Omega z_c(1-\cos\theta) \tag{2.34}$$

The above inequality gives an upper bound for $|g_z|$, which when substituted in (2.30) leads, in conjunction with $1 = \Omega z_c g_{z_c} + \Omega z_c \hat{\Pi}_{z_c}(0)$ and (2.4), to

$$|\hat{F}_z(0)| \geqslant \Omega z_c \left[g_{z_c} \left\{ 1 - \left[1 - \frac{2\Omega z_c}{g_{z_c}^2}(1-\cos\theta) \right]^{1/2} \right\} - K' L^{1-d} \right] \tag{2.35}$$

for some K' which is independent of L. Using (2.4) and (2.27) gives

$$|\hat{F}_z(0)| \geqslant \frac{1}{5} \left[1 - \left(1 - \frac{1-\cos\theta}{40} \right)^{1/2} \right] - O(L^{1-d}) \tag{2.36}$$

The right side is bounded below by a positive constant, independent of sufficiently large L, for $\theta \in [\theta_0, 2\pi - \theta_0]$. ∎

Now, to bound (2.23), we proceed as follows. We first note that by (2.10), (2.4), and Lemmas 2.1 and 2.2,

$$|\hat{h}'_z| \leqslant O(|z_c - z|^{-1/2}) \tag{2.37}$$

This gives the desired bound for the first term on the right side of (2.23). For the second term, we just use (2.8) to obtain $|\hat{h}_{z_c} - \hat{h}_z| \leqslant \text{const} \cdot |z_c - z|^{\varepsilon}$. For the last two terms, it suffices to show that $|\mathscr{E}'_1| \leqslant O(|z_c - z|^{\varepsilon - 3/2})$. In view of (2.19), for this it suffices to show that $|E'| \leqslant O(|z_c - z|^{\varepsilon})$. This last bound follows from (2.15)–(2.17) and Lemma 2.1.

This completes the proof of Theorem 1.1(a), given Lemma 2.1.

2.3. Proof of Theorem 1.1(b) Given Lemma 2.1

In this section we give the proof of Theorem 1.1(b), given Lemma 2.1. Given a function f on $\mathbf{Z}^d$, we define its Fourier transform by

$$\hat{f}(k) = \sum_x f(x)\, e^{ik \cdot x}, \qquad k \in [-\pi, \pi]^d \tag{2.38}$$

whenever the right side makes sense. Then the radius of gyration (1.3) satisfies

$$R_n^2 = -\frac{\nabla_k^2 \hat{t}_n(0)}{2t_n} \tag{2.39}$$

where ∇_k denotes the gradient. Given Theorem 1.1(a), to prove Theorem 1.1(b) it suffices to show that

$$\nabla_k^2 \hat{t}_n(0) = \text{const} \cdot \lambda^n [1 + O(n^{-\varepsilon})] \tag{2.40}$$

The left side is the coefficient of z^n in $\nabla_k^2 \hat{G}_z(0)$, where, by Eq. (3.8) of ref. 1,

$$\hat{G}_z(k) = \frac{\hat{h}_z(k)}{1 - z\,|\Omega|\,\hat{D}(k)\,\hat{h}_z(k)}$$

with

$$\hat{D}(k) = \frac{1}{\Omega} \sum_{\|x\|_\infty \leqslant L,\, x \neq 0} e^{ik \cdot x}$$

Making use of symmetry, a straightforward calculation gives

$$\nabla_k^2 \hat{G}_z(0) = \frac{\Omega z \hat{h}_z(0)^2\, \nabla_k^2 \hat{D}(0) + \nabla_k^2 \hat{\Pi}_z(0)}{\hat{F}_z(0)^2} \tag{2.41}$$

To prove (2.40), by Lemma 1.2 it suffices to show that

$$\nabla_k^2 \hat{G}_z(0) = \frac{\text{const}}{z_c - z} + \mathscr{E}_2(z)$$

with $|\mathscr{E}_2'(z)| \leqslant \text{const} \cdot |z_c - z|^{\varepsilon - 2}$ for all $|z| \leqslant z_c$. This can be shown using (2.41), employing Lemma 2.1 as in Section 2.2. It is at this point that (2.9) and Lemma 2.1(v) are used. This completes the proof of Theorem 1.1(b).

Integrating the bound on $\mathscr{E}_2'$ gives $|\mathscr{E}_2(z)| \leqslant \text{const} \cdot |z_c - z|^{\varepsilon - 1}$, which with (1.12) improves (1.9) to

$$\xi_2(z) = \frac{\text{const}}{(z_c - z)^{1/2}} + O(|z_c - z|^{\varepsilon - 1/2}) \tag{2.42}$$

3. PROOF OF LEMMA 2.1

To complete the proof of Theorem 1.1, it remains to prove Lemma 2.1. In this section we prove Lemma 2.1, given the bounds in Lemma 3.1 below on the quantity $\hat{\Psi}_z(k)$ occurring in the equation

$$\frac{d}{dz}(z\hat{\Pi}_z(k)) = [z\Omega\chi(z)+1]\ \hat{\Psi}_z(k) \tag{3.1}$$

stated in Lemma 2.1(iii). Lemma 3.1 is proved in Section 4.

We restrict attention to spread-out trees for $d>8$. Nearest-neighbor trees in sufficiently high dimensions can be treated similarly. In this and the next section, we use K_i ($i=1, 2,...$) to denote positive constants which are independent of L but may depend on ε or γ of Lemma 2.1. We always take L sufficiently large that the analysis of ref. 1 is valid.

We begin by stating several estimates. In addition to the bounds stated already in (2.4), it is shown in Section 3.2 of ref. 1 that for $|z| \leqslant z_c$

$$|\nabla_k^2 \hat{\Pi}_z(k)| \leqslant K_1 L^{3-d} \tag{3.2}$$

Also, it follows from (2.4) and Lemma 2.2 that for $|z| \leqslant z_c$

$$|\chi(z)| = \left|\frac{\hat{h}_z(0)}{\hat{F}_z(0)}\right| \leqslant K_2 \left|1-\frac{z}{z_c}\right|^{-1/2} \tag{3.3}$$

Arguing as in (3.10)–(3.12) of ref. 1, and using (2.27), for $z \in [(5\Omega)^{-1}, z_c)$ and L sufficiently large we have

$$\hat{F}_z(k) \geqslant K_3[\chi(z)^{-1} + k^2] \geqslant K_4[(1-z/z_c)^{1/2} + k^2] \tag{3.4}$$

and for $z \in [0, (5\Omega)^{-1}]$ we have

$$\hat{F}_z(k) \geqslant K_4(1-z/z_c)^{1/2} \geqslant K_4(1-9/10)^{1/2} \tag{3.5}$$

Combining (2.4) and (3.5) then shows that for $z \in [0, (5\Omega)^{-1}]$,

$$\chi(z) \leqslant K_7 \tag{3.6}$$

The following lemma gives bounds on $\hat{\Psi}_z(k)$. It will be proved in Section 4. For the statement of the lemma we define

$$\eta(x) \equiv \begin{cases} 1, & d>10 \\ (1+|\ln x|)^4, & d=10 \\ x^{(10-d)/2}, & d<10 \end{cases} \tag{3.7}$$

This quantity occurs as an upper bound for a variety of Feynman diagrams with massive propagators that we will encounter. The simplest such diagram is the pentagon diagram

$$\int_{[-\pi,\pi]^d} \frac{1}{(k^2+m^2)^5} \frac{d^dk}{(2\pi)^d} \leqslant O(\eta(m^{-2})) \tag{3.8}$$

(as $m^2 \to 0$); such estimates will be discussed in more detail in Section 4.3.2.

Lemma 3.1. Let $d > 8$. For L sufficiently large (3.1) is satisfied, with $\hat{\Psi}_z(k)$ a power series in z (without constant term) satisfying the following bounds for any z with $0 \leqslant z \leqslant z_c$:

$$\lim_{L\to\infty} \hat{\Psi}_z(k) = 0 \tag{3.9}$$

$$\left\| \frac{d}{dz} \hat{\Psi}_z(0) \right\| \leqslant K \frac{\chi(z)}{z_c} \eta(\chi(z)) \tag{3.10}$$

$$\|\nabla_k^2 \hat{\Psi}_z(0)\| \leqslant KL^2 \eta(\chi(z)) \tag{3.11}$$

In the remainder of this section we give the proof of Lemma 2.1, assuming Lemma 3.1. In preparation for the proof we recall the following identity from Lemma 3.1 of ref. 3. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be a power series with radius of convergence $R > 0$. Then for any z with $|z| \leqslant R$ and for any $\varepsilon \in (0, 1)$,

$$\|\delta_z^\varepsilon f(z)\| = C_{1-\varepsilon} |z| \int_0^\infty \|f'[z \exp(-\lambda^{1/(1-\varepsilon)})]\| \exp(-\lambda^{1/(1-\varepsilon)})\, d\lambda \tag{3.12}$$

where $C_{1-\varepsilon} = [(1-\varepsilon)\, \Gamma(1-\varepsilon)]^{-1}$.

Proof of Lemma 2.1(i). We use (2.4), (2.27), and (3.9) in (2.12), to conclude that

$$\left| 2\hat{F}_z(0) \frac{d\hat{F}_z(0)}{dz} \right| \leqslant K\Omega \tag{3.13}$$

Integrating the above along the line segment connecting z to z_c, using $\hat{F}_{z_c}(0) = 0$, gives

$$|\hat{F}_z(0)|^2 \leqslant K\Omega\, |z - z_c| \leqslant K'\, |1 - z/z_c| \tag{3.14}$$

where we used (2.27) in the last step.

Proof of Lemma 2.1(ii). By (3.12),

$$\|\delta_z^{\gamma}[z\hat{h}_z(0)]\,|_{z_c}\| = C_{1-\gamma} z_c \int_0^{\infty} d\lambda\, \exp(-\lambda^{1/(1-\gamma)}) \left\| \frac{d}{dz}(z\hat{h}_z(0))\,|_{z_c \exp(-\lambda^{1/(1-\gamma)})} \right\| \tag{3.15}$$

By (2.10) and (3.1), the derivative in the integrand satisfies

$$\left\| \frac{d}{dz}(z\hat{h}_z(0)) \right\| \leqslant \chi(|z|) + [|z|\ \Omega\chi(|z|) + 1]\ \|\hat{\Psi}_z(0)\| \tag{3.16}$$

By (3.3) and (3.9), for $z \in (0, z_c)$ the right side can be bounded absolutely to give

$$\left\| \frac{d}{dz}(z\hat{h}_z(0)) \right\| \leqslant K_7 \left(1 - \frac{z}{z_c}\right)^{-1/2} + K_8 \tag{3.17}$$

Substituting the above bound into (3.15), and using (2.27) to bound z_c, (3.15) is bounded absolutely by

$$K_9 \Omega^{-1} \int_0^{\infty} d\lambda\, \exp(-\lambda^{1/(1-\gamma)}) \{ [1 - \exp(-\lambda^{1/(1-\gamma)})]^{-1/2} + 1 \} \tag{3.18}$$

The desired result now follows from the fact that the above integral is finite for $\gamma < 1/2$.

.*Proof of Lemma 2.1*(iii). We take

$$b = z_c \Omega \hat{\Psi}_{z_c}(0), \qquad \beta(z) = z\Omega\hat{\Psi}_z(0) - z_c \Omega \hat{\Psi}_{z_c}(0) \tag{3.19}$$

It then follows immediately from (3.9) that $\lim_{L\to\infty} b = 0$.

To prove the bound on $\beta(z)$, by (2.6) it suffices to prove that

$$\|\delta_z^{\varepsilon}[z\Omega\hat{\Psi}_z(0)]\,|_{z_c}\| < K_{10} \tag{3.20}$$

For this we use (3.12) and the abbreviation

$$z_\lambda = z_c \exp(-\lambda^{1/(1-\varepsilon)}) \tag{3.21}$$

to write

$$\left\| \delta_z^\varepsilon [z\Omega \hat{\Psi}_z(0)] \,|_{z_c} \right\| = C_{1-\varepsilon} z_c \int_0^\infty d\lambda \exp(-\lambda^{1/(1-\varepsilon)}) \left\| \frac{d}{dz} (z\Omega \hat{\Psi}_z(0)) \,|_{z_\lambda} \right\| \tag{3.22}$$

Using (3.10) to estimate the z derivative of $\hat{\Psi}_z(0)$ and then applying (3.3) gives

$$\begin{aligned} \|\delta_z^\varepsilon [z\Omega \hat{\Psi}_z(0)] \,|_{z_c}\| &\leqslant K \int_0^\infty d\lambda \exp(-\lambda^{1/(1-\varepsilon)})[1 + \chi(z_\lambda)\, \eta(\chi(z_\lambda))] \\ &\leqslant K' \int_0^\infty d\lambda \exp(-\lambda^{1/(1-\varepsilon)})[1 + \lambda^{-1/2(1-\varepsilon)} \eta(\lambda^{-1/2(1-\varepsilon)})] \end{aligned} \tag{3.23}$$

The right side of (3.23) is finite for $\varepsilon < 1/2$ if $d > 10$, or for $\varepsilon < (d-8)/4$ if $8 < d \leqslant 10$. In view of (3.9), this completes the proof.

Proof of Lemma 2.1 (iv). Using (3.12), we have

$$\|\delta_z^\varepsilon \nabla_k^2 \hat{\Pi}_{z_c}(0)\| = C_{1-\varepsilon} z_c \int_0^\infty d\lambda \exp(-\lambda^{1/(1-\varepsilon)}) \left\| \frac{d}{dz} \nabla_k^2 \hat{\Pi}_z(0) \,|_{z_\lambda} \right\| \tag{3.24}$$

By (3.1), the derivative in the integrand is given by

$$\nabla_k^2 \frac{d}{dz} \hat{\Pi}_z(0) = z^{-1} [\chi(z)\, z\Omega \nabla_k^2 \hat{\Psi}_z(0) + \nabla_k^2 \hat{\Psi}_z(0) - \nabla_k^2 \hat{\Pi}_z(0)] \tag{3.25}$$

Since $\chi(z) \geqslant 1$, this gives

$$\begin{aligned} \|\delta_z^\varepsilon \nabla_k^2 \hat{\Pi}_{z_c}(0)\| \leqslant C_{1-\varepsilon} z_c \int_0^\infty d\lambda \exp(-\lambda^{1/(1-\varepsilon)})\, \chi(z_\lambda)\, z_\lambda^{-1} [z_\lambda \Omega \, \|\nabla_k^2 \hat{\Psi}_{z_\lambda}(0)\| \\ + \|\nabla_k^2 \hat{\Psi}_{z_\lambda}(0)\| + \|\nabla_k^2 \hat{\Pi}_{z_\lambda}(0)\|] \end{aligned} \tag{3.26}$$

To estimate the integral on the right side, we consider separately the intervals $[0, \lambda_0]$ and $[\lambda_0, \infty)$, where λ_0 is defined by $z_{\lambda_0} = (5\Omega)^{-1}$.

On the interval $[\lambda_0, \infty)$, $z_\lambda \leqslant (5\Omega)^{-1}$, and hence by Lemma 3.1 and (3.6) the contribution to the integral from this range of λ is bounded by L^2 times a constant independent of λ. [The product of z_λ^{-1} with the quantity in brackets in (3.26) can be bounded above by its value at $z_{\lambda_0} = (5\Omega)^{-1}$

using Lemma 3.1; $\hat{\Pi}_z(k)$ also has no z^0 term.] The contribution to the integral due to $\lambda \leqslant \lambda_0$ is bounded above, using (3.2), (3.3), and (3.11), by

$$K_{11} L^2 \int_0^\infty d\lambda \exp(-\lambda^{1/(1-\varepsilon)}) \lambda^{-1/2(1-\varepsilon)} [\eta(\lambda^{-1/2(1-\varepsilon)}) + 1] \tag{3.27}$$

This integral is finite for $\varepsilon < \min\{1/2, (d-8)/4\}$, as in (3.23).

Proof of Lemma 2.1(v). Arguing as in the proof of Lemma 2.1(iv), $(d/dz) \nabla_k^2 \hat{\Pi}_z(0)$ is bounded for large L by a multiple of L^{d+2} uniformly in $|z| \leqslant (5\Omega)^{-1}$. Thus, it suffices to consider $|z| \in [(5\Omega)^{-1}, z_c)$. By (3.25), (3.3), and (3.11), for this range of z we have

$$\left\| \frac{d}{dz} \nabla_k^2 \hat{\Pi}_z(0) \right\| \leqslant K_{12} L^{2+d} \left| 1 - \frac{z}{z_c} \right|^{-1/2} \left[\eta \left(\left| 1 - \frac{z}{z_c} \right|^{-1/2} \right) + 1 \right] \tag{3.28}$$

The desired bound then follows from the definition of η.

4. PROOF OF LEMMA 3.1

In this section we prove Lemma 3.1, thus completing the proof of Theorem 1.1. We begin in Section 4.1 by obtaining an explicit expression for $\hat{\Psi}_z(k)$, then obtain bounds on this expression in terms of Feynman diagrams in Section 4.2, and finally in Section 4.3 obtain bounds on the Feynman diagrams. New technical difficulties, not previously encountered in lace expansion analyses, arise both in the derivation of the expression for $\hat{\Psi}_z(k)$ and in the Feynman diagram bounds.

4.1. An Expression for $\hat{\Psi}_z(k)$

In this section we obtain an explicit expression for the quantity $\hat{\Psi}_z(k)$ in the equation

$$\frac{d}{dz} (z \hat{\Pi}_z(k)) = [z\Omega\chi(z) + 1] \, \hat{\Psi}_z(k) \tag{4.1}$$

The basic idea is roughly as follows. Consider the N-loop contribution to $\hat{\Pi}_z(k)$. Multiplying by z gives an overall factor of z raised to a power equal to the number of sites, which when differentiated gives a factor equal to the number of sites. This factor can be replaced by a sum over sites y. Diagrammatically this corresponds to introducing a new line emanating from the N-loop diagram and terminating at y, which is interacting with the rest of the diagram. If we were to ignore this interaction, as would be appropriate for an upper bound, we would obtain an overall factor of $\chi(z)$ multiplied by a sum of N-loop diagrams having an extra vertex. These last diagrams will be finite in more than eight dimensions, and are closely related to $\hat{\Psi}_z$. However, here we are interested in obtaining an *identity*, not just an upper bound, and so, to extract the factor of $\chi(z)$, we perform an expansion to remove the interaction between the new line terminating at y and the rest of the diagram. This introduces higher-order diagrams, which must eventually be bounded.

We follow the notation of ref. 1 without further mention. We also introduce the following definitions. Given $a<b$, and a function $\mathscr{X}_{st}$ defined on pairs s, t with $a \leqslant s < t \leqslant b$, we write

$$K[a, b; \mathscr{X}] = \prod_{a \leqslant s < t \leqslant b} (1 + \mathscr{X}_{st}) \tag{4.2}$$

and

$$J[a, b; \mathscr{X}] = \sum_{L \in \mathscr{L}[a,b]} \prod_{st \in L} \mathscr{X}_{st} \prod_{s't' \in \mathscr{C}(L)} (1 + \mathscr{X}_{s't'}) \tag{4.3}$$

We adopt the usual convention that an epty product is equal to unity, so that in particular $K[b+1, b; \mathscr{X}] = K[b, b; \mathscr{X}] = 1$. We also define $J[a, a; \mathscr{X}] = 1$.

We begin from the definition of $\Pi_z(0, x)$ in Eq. (2.10) of ref. 1:

$$\Pi_z(0, x) = \sum_{\substack{\omega: 0 \to x \\ |\omega| \geqslant 1}} z^{|\omega|} \left(\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right) J[0, |\omega|; \mathscr{U}] \tag{4.4}$$

where the sum is over ordinary (not necessarily self-avoiding) spread-out walks ω. Differentiating gives

$$\frac{d}{dz}(z\Pi_z(0, x)) = \sum_y \sum_{\substack{\omega: 0 \to x \\ |\omega| \geqslant 1}} z^{|\omega|} \sum_{j=0}^{|\omega|} \left(\prod_{i:\, i \neq j} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right) \times \sum_{R_j \ni \omega(j),\, y} z^{|R_j|} J[0, |\omega|; \mathscr{U}] \tag{4.5}$$

To simplify the notation, we will write

$$\gamma_i = \sum_{R_i \ni \omega(i)} z^{|R_i|} \tag{4.6}$$

with the understanding that the summation on the right side is not yet closed and may also act on other factors [such as $J[0, |\omega|; \mathscr{U}]$ in (4.5)].

Given a rib R_j containing y and $\omega(j)$, we denote its backbone from $\omega(j)$ to y by ω', and consider R_j to be composed of ω' and ribs attached to ω'. We denote the rib containing $\omega'(k)$ by R'_k, so that R_j is the disjoint union, as sets of bonds, of ω', R'_0, R'_1,..., $R'_{|\omega'|}$. We use γ'_i to denote the analogue of (4.6) with R_i replaced by R'_i and $\omega(i)$ by $\omega'(i)$. We introduce an interaction

$$\mathscr{U}'_{kl} = \begin{cases} 0 & \text{if } \; R'_k \cap R'_l = \varnothing \\ -1 & \text{if } \; R'_k \cap R'_l \neq \varnothing \end{cases} \tag{4.7}$$

between the prime ribs, so that

$$\sum_{R_j \ni \omega(j),\, y} z^{|R_j|} = \sum_{\omega' : \omega(j) \to y} z^{|\omega'|} \left(\prod_{k=0}^{|\omega'|} \gamma'_k \right) K[0, |\omega'|; \mathscr{U}'] \tag{4.8}$$

where the sum is over ordinary walks.

Using the convention that ts represents the edge st if $t > s$, for fixed $j \in \{0, 1,..., |\omega|\}$ we have

$$J[0, |\omega|; \mathscr{U}] = \sum_{L \in \mathscr{L}[0, |\omega|]} \mathscr{U}(L, j) \left(\prod_{s:\, sj \in L} \mathscr{U}_{sj} \prod_{t:\, tj \in \mathscr{C}(L)} (1 + \mathscr{U}_{tj}) \right) \tag{4.9}$$

where

$$\mathscr{U}(L, j) = \prod_{\substack{st \in L \\ s, t \neq j}} \mathscr{U}_{st} \prod_{\substack{s't' \in \mathscr{C}(L) \\ s', t' \neq j}} (1 + \mathscr{U}_{s't'}) \tag{4.10}$$

represents the interaction among ribs other than R_j. The interaction between the prime and original ribs is given, for $k \neq j$, by

$$\mathscr{V}'_{kl} = \begin{cases} 0 & \text{if } \; R_k \cap R'_l = \varnothing \\ -1 & \text{if } \; R_k \cap R'_l \neq \varnothing \end{cases} \tag{4.11}$$

Let

$$1 + \mathscr{V}'_l = \prod_{s:\, sj \in \mathscr{C}(L)} (1 + \mathscr{V}'_{sl}) \tag{4.12}$$

and

$$1 + \mathscr{X}_{kl} = \begin{cases} 1 + \mathscr{U}'_{kl}, & 0 < k < l \\ (1 + \mathscr{U}'_{0l})(1 + \mathscr{V}'_l), & 0 = k < l \end{cases} \tag{4.13}$$

For $0 < k < l$, $\mathscr{X}_{kl}$ involves only interactions between the prime ribs, while interactions between prime and original ribs are encoded in $\mathscr{X}_{0l}$. Then we have

$$K[0, |\omega'|; \mathscr{U}'] \prod_{t:\, tj \in \mathscr{C}(L)} (1+\mathscr{U}_{tj}) = (1+\mathscr{V}'_0) \prod_{0 \leqslant k < l \leqslant |\omega'|} (1+\mathscr{X}_{kl})$$
$$= (1+\mathscr{V}'_0)\, K[0, |\omega'|; \mathscr{X}] \tag{4.14}$$

and (4.5) can be written

$$\frac{d}{dz}(z\Pi_z(0, x)) = \sum_y \sum_{\substack{\omega:\, 0 \to x \\ |\omega| \geqslant 1}} z^{|\omega|} \sum_{j=0}^{|\omega|} \left(\prod_{i:\, i \neq j} \gamma_i \right) \sum_{L \in \mathscr{L}[0, |\omega|]} \mathscr{U}(L, j)$$
$$\times \sum_{\omega':\, \omega(j) \to y} z^{|\omega'|} \left(\prod_{k=0}^{|\omega'|} \gamma'_k \right) (1+\mathscr{V}'_0)\, K[0, |\omega'|; \mathscr{X}] \prod_{s:\, sj \in L} \mathscr{U}_{sj} \tag{4.15}$$

We wish now to perform the lace expansion, by expanding and resumming the interaction factor $K[0, |\omega'|; \mathscr{X}]$. First we extract the contribution due to the simplest case, when $|\omega'| = 0$. In this case we have the constraint $y = \omega(j)$, which corresponds in (4.15) to a sum over all sites on the original backbone ω. This gives a contribution to (4.15) equal to

$$\Psi_2(0, x) = \sum_y \Psi_2(0, x, y) \tag{4.16}$$

where

$$\Psi_2(0, x, y) = \sum_{\substack{\omega:\, 0 \to x \\ |\omega| \geqslant 1}} I[y \in \omega]\, z^{|\omega|} \left(\prod_{i=0}^{|\omega|} \gamma_i \right) J[0, |\omega|] \tag{4.17}$$

For $|\omega'| \geqslant 1$ we distinguish between the case where j is an endpoint of an edge in the lace L and the case where it is not. In the latter case the empty product $\prod_{s:\, sj \in L} \mathscr{U}_{sj} = 1$ can be ignored, and we will insert Eq. (2.6) of ref. 1, i.e.,

$$K[0, |\omega'|; \mathscr{X}] = K[1, |\omega'|; \mathscr{X}] + \sum_{a=1}^{|\omega'|} J[0, a; \mathscr{X}]\, K[a+1, |\omega'|; \mathscr{X}] \tag{4.18}$$

into (4.15). By definition of $\mathscr{X}$, it follows from (4.18) that

$$K[0, |\omega'|; \mathscr{X}] = K[1, |\omega'|; \mathscr{U}'] + \sum_{a=1}^{|\omega'|} J[0, a; \mathscr{X}]\, K[a+1, |\omega'|; \mathscr{U}'] \tag{4.19}$$

Next we consider the case where L contains an edge with endpoint j. For brevity we write in this case $j \in L$. By definition of a lace there can be at most two edges in any lace L containing a particular site, so in this case there will be either one or two edges in L containing j. Given an edge $sj \in L$ and a configuration for which $\mathscr{U}_{sj} = -1$, let $k(s, j)$ be the smallest k such that $R_s \cap R'_k \neq \varnothing$, and set $k(L, j) = \max_{s:\, sj \in L} k(s, j)$. Then in (4.15) we write

$$\prod_{s:\, sj \in L} \mathscr{U}_{sj} = \pm \sum_{l=0}^{|\omega'|} I[k(L, j) = l] \equiv \sum_{l=0}^{|\omega'|} \mathscr{W}_l \tag{4.20}$$

where the last equivalence defines $\mathscr{W}_l$. In the above, the sign $\pm$ is taken to be $+1$ if there are two edges $sj \in L$, and to be -1 when there is only one $sj \in L$; this sign will be irrelevant for our absolute bounds. Now we wish to perform the lace expansion, this time taking into account the extra constraint due to $\mathscr{W}_l$ on the sums over ω' and R'_k. We reorganize the interactions before performing the lace expansion, as follows. Fix $0 \leqslant l < |\omega'|$, and let

$$1 + \mathscr{Y}^{(l)}_{st} = \begin{cases} 1 + \mathscr{U}'_{st}, & l < s < t \\ \displaystyle\prod_{k=0}^{l} (1 + \mathscr{X}_{kt}), & l = s < t \end{cases} \tag{4.21}$$

Then for $0 \leqslant l \leqslant |\omega'|$,

$$K[0, |\omega'|; \mathscr{X}] = K[0, l; \mathscr{X}]\, K[l, |\omega'|; \mathscr{Y}^{(l)}] \tag{4.22}$$

(recall that $K[b, b; \mathscr{Y}^{(l)}] = 1$). By Eq. (2.6) of ref. 1, for $0 \leqslant l \leqslant |\omega'|$,

$$K[l, |\omega'|; \mathscr{Y}^{(l)}] = K[l+1, |\omega'|; \mathscr{U}'] + \sum_{a=l+1}^{|\omega'|} J[l, a; \mathscr{Y}^{(l)}]\, K[a+1, |\omega'|; \mathscr{U}'] \tag{4.23}$$

where for $l = |\omega'|$ the empty sum over a is taken to be zero.

To abbreviate the notation, we write the bulk of the right side of (4.15) as

$$\begin{aligned} \mathscr{F} &\equiv \sum_y \mathscr{F}_y \\ &\equiv \sum_y \sum_{\substack{\omega:\, 0 \to x \\ |\omega| \geqslant 1}} z^{|\omega|} \sum_{j=0}^{|\omega|} \left(\prod_{i:\, i \neq j} \gamma_i \right) \sum_{L \in \mathscr{L}[0, |\omega|]} \mathscr{U}(L, j) \\ &\quad \times \sum_{\substack{\omega':\, \omega(j) \to y \\ |\omega'| \geqslant 1}} z^{|\omega'|} \left(\prod_{k=0}^{|\omega'|} \gamma'_k \right) (1 + \mathscr{V}'_0) \end{aligned} \tag{4.24}$$

with the understanding that the summations on the right side are not yet closed. Extracting the $|\omega'|=0$ term from the right side of (4.15) and using (4.19), (4.22), and (4.23) gives

$$\begin{aligned}
&\frac{d}{dz}(z\Pi_z(0,x)) \\
&\quad = \Psi_2(0,x) + \mathscr{F}\Bigg\{ I[j \notin L]\Bigg(K[1,|\omega'|;\mathscr{U}'] \\
&\qquad + \sum_{a=1}^{|\omega'|} J[0,a;\mathscr{X}]\, K[a+1,|\omega'|;\mathscr{U}'] \Bigg) \\
&\qquad + I[j \in L] \sum_{l=0}^{|\omega'|} \mathscr{W}_l K[0,l;\mathscr{X}] \\
&\qquad \times \Bigg(K[l+1,|\omega'|;\mathscr{U}'] + \sum_{a=l+1}^{|\omega'|} J[l,a;\mathscr{Y}^{(l)}]\, K[a+1,|\omega'|;\mathscr{U}'] \Bigg) \Bigg\}
\end{aligned} \tag{4.25}$$

Let D equal Ω^{-1} times the indicator function for the set of sites $x \neq 0$ such that $\|x\|_\infty \leqslant L$. Then resummation as in the derivation of Eq. (2.11) of ref. 1 gives

$$\begin{aligned}
&\mathscr{F}(I[j \notin L]\, K[1,|\omega'|;\mathscr{U}'] + I[j \in L]\, \mathscr{W}_0 K[1,|\omega'|;\mathscr{U}']) \\
&\quad = \sum_y \sum_u \Psi_2(0,x,u)(z\Omega D * G_z)(y-u) \\
&\quad = \Psi_2(0,x)\, z\Omega\chi(z)
\end{aligned} \tag{4.26}$$

This gives

$$\begin{aligned}
\frac{d}{dz}(z\Pi_z(0,x)) = {}& \Psi_2(0,x)[1 + z\Omega\chi(z)] \\
& + \mathscr{F}\Bigg\{ I[j \notin L] \sum_{a=1}^{|\omega'|} J[0,a;\mathscr{X}]\, K[a+1,|\omega'|;\mathscr{U}'] \\
& + I[j \in L] \Bigg(\sum_{l=1}^{|\omega'|} \mathscr{W}_l K[0,l;\mathscr{X}]\, K[l+1,|\omega'|;\mathscr{U}'] \\
& + \sum_{l=0}^{|\omega'|-1} \mathscr{W}_l K[0,l;\mathscr{X}] \\
& \times \sum_{a=l+1}^{|\omega'|} J[l,a;\mathscr{Y}^{(l)}]\, K[a+1,|\omega'|;\mathscr{U}'] \Bigg) \Bigg\}
\end{aligned} \tag{4.27}$$

Now we define

$$\Psi_1(0, x, y) = \mathscr{F}_y \Big\{ I[j \notin L]\, J[0, |\omega'|; \mathscr{X}] + I[j \in L] \sum_{l=0}^{|\omega'|} \mathscr{W}_l K[0, l; \mathscr{X}]\, J[l, |\omega'|; \mathscr{Y}^{(l)}] \Big\} \tag{4.28}$$

and

$$\Psi_1(0, x) = \sum_y \Psi_1(0, x, y) \tag{4.29}$$

The quantity $\Psi_1(0, x)$ incorporates from (4.27) the $a = |\omega'|$ term of the first summation, the $l = |\omega'|$ term of the second summation, and the $a = |\omega'|$ term of the fourth summation. The rest of the term involving $\mathscr{F}$ in (4.27) can be resummed to give

$$\sum_y \sum_u \Psi_1(0, x, u)(z\Omega D * G_z)(y-u) = \Psi_1(0, x)\, z\Omega\chi(z) \tag{4.30}$$

Writing

$$\Psi(0, x) = \Psi_1(0, x) + \Psi_2(0, x) \tag{4.31}$$

we finally have

$$\frac{d}{dz}(z\Pi_z(0, x)) = \Psi(0, x)[1 + z\Omega\chi(z)] \tag{4.32}$$

Taking the Fourier transform then gives

$$\frac{d}{dz}(z\hat{\Pi}_z(k)) = [1 + z\Omega\chi(z)]\, \hat{\Psi}_z(k) \tag{4.33}$$

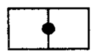
By definition, $\hat{\Psi}(k)$ is a power series in z without constant term.

4.2. Bounds in Terms of Feynman Diagrams

In the previous section we obtained an explicit expression for $\hat{\Psi}(k)$ in which it was written as a sum of two terms $\hat{\Psi}_1(k)$ and $\hat{\Psi}_2(k)$. In this section we obtain upper bounds on these two quantities and their derivatives in terms of Feynman diagrams. Then in the next section we show how these diagrams can be bounded to give the results stated in Lemma 3.1. As in the bounds of ref. 1, $\hat{\Psi}(k)$ will be bounded in terms of sums of Feynman diagrams of an increasing number of loops, and these sums will be bounded above by (essentially) geometric series whose ratio is the square diagram. However, here a number of new diagrams will be encountered, not occurring in ref. 1, and these will require individual attention.

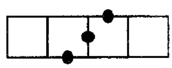
We begin with $\hat{\Psi}_2(k)$, which is the simpler of the two.

4.2.1. Bounds on $\hat{\Psi}_2(k)$. From (4.17) it is apparent that the diagrams contributing to $\hat{\Psi}_2(k)$ are given by the diagrams obtained by adding a vertex to each line along the backbone of the diagrams contributing to $\hat{\Pi}(k)$. The latter are illustrated in Fig. 3 of ref. 1. By neglecting interactions between distinct lines, these diagrams with additional vertex can be bounded as products of the square diagram and the diagram



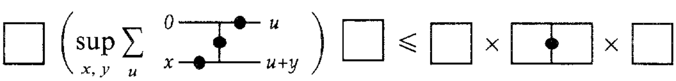
(4.34)

(heavy dots on diagram lines represent vertices). For example, the diagram



(4.35)

can be bounded above by



(4.36)

To prove the first statement of Lemma 3.1 for $\hat{\Psi}_2(k)$, it thus suffices to show that at $z = z_c$ the diagram (4.34) goes to zero as $L \to \infty$.

The z derivative of $z\hat{\Psi}_2(k)$ can be bounded diagrammatically as explained at the beginning of the previous section. That is, $z(d/dz)\,\hat{\Psi}_2(k)$ can be represented by the small quantity $-\hat{\Psi}_2(k)$, plus terms given by the diagrams of $\hat{\Psi}_2(k)$ with an extra vertex and a line emanating from this vertex which is interacting with the rest of the diagram. We ignore this interaction and take absolute values in each diagram, to get an upper bound for z positive of the form

$$\left\| \frac{d}{dz}\,\hat{\Psi}_2(k) \right\| \leqslant \frac{1}{z}\,\Phi_z \chi(z) + \frac{1}{z}\,\|\hat{\Psi}_2(k)\| \tag{4.37}$$

Here, Φ_z is given by the sum of diagrams which are obtained from the diagrams of $\hat{\Psi}_2$ by adding an extra vertex, or in other words by the sum of the diagrams obtained by adding two additional vertices to the diagrams of $\hat{\Pi}(k)$. Note that by construction Φ_z is a power series in z with positive coefficients, without constant term.

Given such a diagram, an upper bound is obtained by neglecting interactions between distinct lines. For $0 \leqslant z \leqslant (5\Omega)^{-1}$, an upper bound on $z^{-1}\Phi_z$ is given by its value at $z=(5\Omega)^{-1}$, since it is a power series in z with positive coefficients. The diagrams in Φ_z are then bounded by constants uniformly in L, since by (3.6) the susceptibility is bounded by a constant. This gives a bound on (4.37), of order z_c^{-1}, for $0 \leqslant z \leqslant (5\Omega)^{-1}$. Now for $(5\Omega)^{-1} \leqslant z \leqslant z_c$ we again bound the factor z^{-1} from above by $5\Omega = O(z_c^{-1})$. The diagrams in Φ_z can be bounded by following the procedure used in the previous paragraph, with the difference that now we may pull off a pentagon diagram rather than a square diagram. The bounds will then be in terms of the square diagram and the additional diagrams

 (4.38)

It suffices to show that these additional diagrams are bounded above by a multiple of $\eta(\chi(z))$.

Finally, for the bound on $\sum_x |x|^2\, \Psi_2(0, x)$ we distribute the $|x|^2$ using the triangle inequality to obtain diagrams in which a single line is weighted by $|x|^2$. Then we use the fact that the Fourier transform of $|x|^2\, G_z(0, x)$ is bounded above by

$$|\nabla_k^2 \hat{G}_z(k)| \leqslant K[G_z(k) + L^2 G_z(k)^2] \tag{4.39}$$

which follows from Eq. (3.22) of ref. 1 and accompanying bounds. This implies that $\nabla_k^2 \hat{\Psi}_2(k)$ can be bounded above by diagrams already encountered in this section.

To summarize, to prove the bounds of Lemma 3.1 for $\hat{\Psi}_2(k)$, it suffices to show that the diagram (4.34) is finite and goes to zero as $L \to \infty$ and that the diagrams (4.38) are bounded above by a multiple of $\eta(\chi)$.

4.2.2. Bounds on $\hat{\Psi}_1(k)$. The quantity $\hat{\Psi}_1(k)$ is given by (4.28) and (4.24). In this section upper bounds will be obtained in terms of convergent diagrams which go to zero as $L \to \infty$. Just as for $\hat{\Psi}_2(k)$, to obtain the bounds of Lemma 3.1 on derivatives of $\hat{\Psi}_1(k)$ it suffices to show that adding a vertex to these convergent diagrams gives rise to a result which is bounded above by a multiple of $\eta(\chi)$. The identification of the diagrams which arise is tedious but straightforward, and much of the detailed diagrammatic analysis will be omitted.

We deal separately with the contribution to (4.28) due to terms with $j \notin L$ and $j \in L$, beginning with the former.

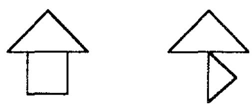
Case 1: $j \notin L$. Suppose $j \notin L$. Consider the factor $J[0, |\omega'|; \mathscr{X}]$ in (4.28) to be a sum over laces L'. For a nonzero contribution to Ψ_1 in this case there are rib intersections imposed by both of the laces L and L'. Roughly speaking, these intersections give rise to a ladder diagram as for $\hat{\Pi}$ with an additional ladder diagram growing from it. The coupling between these two ladder diagrams is mediated by the first edge $0l$ (say) of the lace L', which gives rise to a factor $\mathscr{X}_{0l}$. According to (4.13), this factor will be nonzero only when either $R'_0 \cap R'_l \neq \varnothing$ or $R_s \cap R'_l \neq \varnothing$ for some s with $sj \in \mathscr{C}(L)$.

We begin by looking in detail at the simplest case, in which both laces consist of a single bond. The contribution to $\Psi_1(0, x)$ due to this case is

$$\sum_{y} \sum_{\substack{\omega: 0 \to x \\ |\omega| \geq 1}} z^{|\omega|} \sum_{j=0}^{|\omega|} \left(\prod_{i:\, i \neq j} \gamma_i \right) \mathscr{U}_{0\,|\omega|} \prod_{\substack{0 \leq s < t \leq |\omega| \\ s,t \neq j;\, st \neq 0\,|\omega|}} (1 + \mathscr{U}_{st})$$

$$\times \sum_{\substack{\omega': \omega(j) \to y \\ |\omega'| \geq 1}} z^{|\omega'|} \left(\prod_{k=0}^{|\omega'|} \gamma'_k \right) (1 + \mathscr{V}'_0)\, \mathscr{X}_{0\,|\omega'|} \prod_{\substack{0 \leq s' < t' \leq |\omega| \\ s't' \neq 0\,|\omega'|}} (1 + \mathscr{X}_{s't'}) \qquad (4.40)$$

The factor $\mathscr{X}_{0,\,|\omega'|}$ is nonzero only if $R'_{|\omega'|}$ intersects either some R_s $(0 \leq s \leq |\omega|,\ s \neq j$; in which case $\mathscr{V}'_{|\omega'|} = -1)$, or else R'_0 (in which case $\mathscr{U}'_{0\,|\omega'|} = -1$). Also, the factor $\mathscr{U}_{0\,|\omega|}$ requires an intersection of the ribs R_0 and $R_{|\omega|}$. This leads to the diagrams



(4.41)

which can be bounded easily using the square diagram.

Consider now the case where the laces L and L' may consist of more than one edge. By definition of $\mathscr{X}$, edges in L' other than the first force intersections between prime ribs and do not involve unprimed ribs. For this reason, in diagrammatic estimates we can bound the portion of the diagram resulting from the intersections imposed by edges in L' other than the first, in terms of the square diagram in the usual manner. Similarly, since the first edge in L' can impose only intersections between prime ribs and unprimed ribs compatible with R_j (or between a prime rib and R'_0), we can bound loops due to L which do not overlap with these compatible ribs also in the usual way. Some examples of situations that can occur, together with the diagrams arising, are illustrated in Fig. 1. Additional diagrams can arise, and are depicted in Fig. 4. These possibilities have been arrived at by a tedious case-by-case analysis which we omit.

Thus, for this case it suffices to show that the diagrams of Fig. 4 converge and go to zero as $L \to \infty$, and that the diagrams obtained by adding an additional vertex to these diagrams are bounded above by a multiple of $\eta(\chi)$.

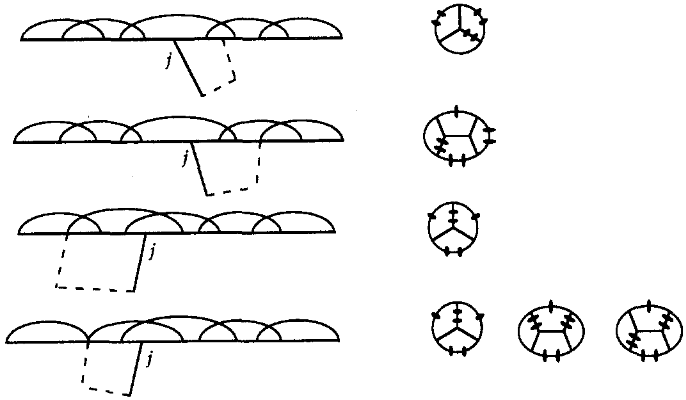


Fig. 1. Four representative partial laces L with $j \notin L$, and the intersections imposed by $\mathscr{X}_{0l}$, together with the diagrams to which they give rise.

Case 2: $j \in L$. In this case, in (4.28) the value of j for which the rib R_j contains y is an endpoint of one or two edges of the lace L. We proceed as in the previous case, this time taking into account the additional constraint $\mathscr{W}_l$ [defined in (4.20)]. Again in bounding diagrams we need only consider the first edge in the lace L' arising from $J[l, |\omega'|; \mathscr{Y}^{(l)}]$. This is because for $t > s > l$ the interaction $\mathscr{Y}^{(l)}_{st}$ involves only the prime ribs, and the factor $\mathscr{W}_l$ does not place any constraint on prime ribs R'_k with $k > l$. We consider separately the cases in which j is an endpoint of one or two edges in L.

Consider first the case where j is an endpoint of a single edge sj of L. In this case the factor $\mathscr{W}_l$ in (4.28) enforces an intersection between R_s and R'_l. Suppose that the first edge of L' is lk. The factor $\mathscr{Y}^{(l)}_{lk}$ in $J[l, |\omega'|; \mathscr{Y}^{(l)}]$ is nonzero if and only if $\mathscr{X}_{sk} \neq 0$ for some $s = 0,..., l$. This occurs if either R'_k intersects R'_s for some $s = 0,..., l$ or if R'_k intersects R_a for some a such that $aj \in \mathscr{C}(L)$. Taking this into account, we are led as in case 1 to a number of

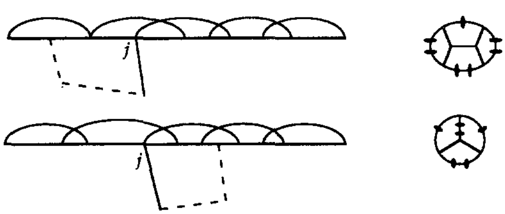


Fig. 2. Two representative laces L with j the endpoint of a single edge in L, together with the intersection imposed by the first edge of L', and the corresponding diagrams.

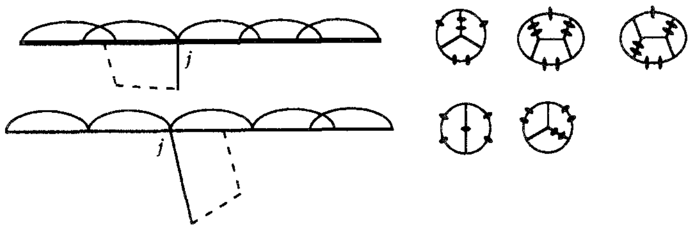


Fig. 3. Two laces L with j the endpoint of two edges in L, together with the intersection imposed by the first edge of L', and the corresponding diagrams.

new diagrams, some of which are indicated in Fig. 2. Other possibilities arise, but no new diagrams are required beyond those illustrated in Fig. 4. We omit the tedious details of the case-by-case analysis showing that no other diagrams are encountered.

For the case where j is an endpoint of a single edge of L it suffices to show that the diagrams of Fig. 4 converge and go to zero as $L \to \infty$, and that the diagrams obtained by adding an additional vertex to these diagrams are bounded above by a multiple of $\eta(\chi)$.

The situation for the case where j is an endpoint of two edges sj and jt of L is similar. Bounds are required on the diagrams of Fig. 4 as outlined in the previous paragraph. Some examples of how the diagrams arise are shown in Fig. 3.

The situation is summarized in Fig. 4. In the next section estimates on the diagrams of Fig. 4 will be obtained.

4.3. Bounds on Feynman Diagrams

In this section we complete the proof of Lemma 3.1 by showing that for $d > 8$ and L sufficiently large the 11 diagrams of Fig. 4 are finite at $z = z_c$ and go to zero as $L \to \infty$, and that the diagrams obtained from these 11 by adding one vertex are bounded above by a multiple of $\eta(\chi)$. We

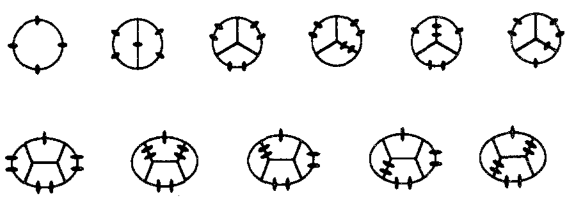


Fig. 4. To complete the proof of Lemma 3.1, it suffices to show that the 11 diagrams depicted converge and go to zero as $L \to \infty$, and that the diagrams obtained by adding a vertex to these diagrams are bounded above by a multiple of $\eta(\chi)$.

822/67/5-6-13

begin by considering the 11 basic diagrams and then move on to the diagrams obtained from these by adding a vertex. Apart from the square diagram, the diagrams of Fig. 4 are basic in the sense that they do not decompose into products of simpler diagrams which are convergent above eight dimensions.

4.3.1. The Eleven Basic Diagrams. For the 11 diagrams depicted in Fig. 4 we set $z = z_c$ and use the infrared bound (3.4) to bound the propagator above by a multiple of k^{-2}. The superficial infrared degree of divergence (d times the number of loops minus twice the number of lines) of an l-loop diagram is equal to $l(d-8)$, so the diagrams will converge unless there is a divergent subdiagram. A theorem of Riesz (Theorem 1 of ref. 8) states that the conventional power counting condition for subgraphs suffices to prove convergence. Thus, we need only check that the infrared degree of divergence of each subgraph is positive. The infrared degree of divergence of a subgraph is defined in ref. 8 essentially as d times the number of loops minus twice the number of "relevant" internal lines of the subgraph. Roughly speaking, a line is relevant for a subgraph if it is not shared by a loop not in the subgraph—we refer the reader to ref. 8 for the precise definition. It is a tedious but routine exercise to check that the required condition is satisfied for each of the 11 basic diagrams, and hence they are all finite.

To see that the 11 basic diagrams go to zero as $L \to \infty$, we first note that by ref. 1, (3.14), and ref. 9, (5.30), for $k \neq 0$

$$\lim_{L \to \infty} \hat{G}_z(k) = g_z \tag{4.42}$$

Since the diagrams are bounded above by their values at z_c, it suffices to consider $z = z_c$. By (4.42) and the dominated convergence theorem, the ten diagrams without the constraint that loops not shrink to a point converge to $g_{z_c}^I$, where I is the number of lines in the diagram. Since this is the contribution to the diagram when all vertices coincide, the diagram with loops constrained to be nontrivial must converge to zero as $L \to \infty$.

4.3.2. The Eleven Diagrams with Added Vertex. The 11 diagrams with added vertex cannot be expected to be finite at the critical point for all $d > 8$, as the superficial infrared degree of divergence $l(d-8)-2$ is negative for d near 8. However, for $d > 10$ this is positive and it can be seen as above that for $d > 10$ the infrared degree of divergence of each subdiagram is positive, and hence by Theorem 1 of ref. 8 the diagrams are all finite.

For $d \leqslant 10$ the manner of divergence of the pentagon diagram can be estimated as follows. By (3.4), for z less than but near z_c the pentagon diagram is bounded above by a multiple of

$$\int_{[-\pi,\pi]^d} \frac{1}{[\chi(z)^{-1}+k^2]^5} \frac{d^dk}{(2\pi)^d} \tag{4.43}$$

By making the change of variables $k' = k\chi(z)^{1/2}$, it is easily seen that this integral is order

$$\begin{array}{ll} 1 & (d>10) \\ 1+\ln\chi & (d=10) \\ \chi^{(10-d)/2} & (d<10) \end{array}$$

and hence [comparing (3.7)] is order $\eta(\chi)$.

To see that the remaining ten diagrams are also order $\eta(\chi)$, we will appeal to Theorem 2 of ref. 8. In preparation for this, we again scale all momenta in the integral representing a diagram as was done for the pentagon. This results in an overall factor of $\chi(z)^{-l(d-8)/2+1}$ multiplied by the same diagram with propagators $[(k')^2+1]^{-1}$ and integrations ranging over $[-\pi\chi^{1/2}, \pi\chi^{1/2}]^d$. We denote this latter diagram by $\mathscr{H}(l, d, \chi)$. A diagram $\mathscr{H}(l, d, \chi)$ is infrared convergent, but may be ultraviolet divergent as $\chi\to\infty$. The subdiagrams giving the maximum ultraviolet degree of divergence (dl minus twice the number of lines of the subdiagram) are the full diagrams with ultraviolet degree $l(d-8)-2$. Hence, by Theorem 2 of ref. 8, we have upper bounds given by multiples of

$$\mathscr{H}(l, d, \chi) \leqslant \begin{cases} 1, & d<8+2/l \\ \chi^{l(d-8)/2-1}(1+\log\chi)^l, & d\geqslant 8+2/l \end{cases} \tag{4.44}$$

Multiplying by the overall factor $\chi(z)^{-l(d-8)/2+1}$ then gives an upper bound for the original diagram of the form

$$\begin{array}{ll} \chi^{l(8-d)/2+1} & (d<8+2/l) \\ (1+\log\chi)^l & (d\geqslant 8+2/l) \end{array} \tag{4.45}$$

If $d=10$, then $d\geqslant 8+2/l$. Thus, we have a bound $(1+\log\chi)^4$ for all diagrams. If $d=9$, then $d<8+2/l$ only for the one-loop diagram, which is then bounded above by order $\chi^{1/2}$. Higher-loop diagrams are bounded by powers of logarithms of χ and hence are also bounded by order $\chi^{1/2}$. This completes the proof.

ACKNOWLEDGMENTS

G.S. is grateful to Hal Tasaki for his hospitality during a visit to Gakushuin University, where part of this work was done. Part of this work was done while T.H. was at the Department of Physics, Gakushuin University. The work of G.S. was supported by NSERC grant A9351.

REFERENCES

1. T. Hara and G. Slade, On the upper critical dimension of lattice trees and lattice animals, *J. Stat. Phys.* **59**:1469–1510 (1990).
2. D. C. Brydges and T. Spencer, Self-avoiding walk in 5 or more dimensions, *Commun. Math. Phys.* **97**:125–148 (1985).
3. T. Hara and G. Slade, Self-avoiding walk in five or more dimensions. I. The critical behaviour. To appear in *Commun. Math. Phys.* (1992).
4. D. J. Klein, Rigorous results for branched polymer models with excluded volume, *J. Chem. Phys.* **75**:5186–5189 (1981).
5. T. C. Lubensky and J. Isaacson, Statistics of lattice animals and dilute branched polymers, *Phys. Rev. A* **20**:2130–2146 (1979).
6. E. J. Janse van Rensberg, On the number of trees in Z^d, Preprint (1992).
7. E. J. J. van Rensberg and N. Madras, A non-local Monte Carlo algorithm for lattice trees, *J. Phys. A: Math. Gen.* **25**:303–333 (1992).
8. T. Riesz, A convergence theorem for lattice Feynman integrals with massless propagators, *Commun. Math. Phys.* **116**:573–606 (1988).
9. T. Hara and G. Slade, Mean-field critical behaviour for percolation in high dimensions, *Commun. Math. Phys.* **128**:333–391 (1990).